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# DIVISOR CAYLEY GRAPHS: A STUDY ON MATCHING NUMBR

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**Abstract:** Let $n \geq 1$ be an integer and let $S$ be the set of divisors of $n$ other than. Then the set $S^* = \{s, n - s \,/\, s \in S\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo $n$. The Cayley graph of $(Z_n, \oplus)$ associated with the above symmetric subset $S^*$ is called the divisor Cayley graph and it is denoted by $G\ (Zn, D)$. In this paper we study the edge cover, edge covering number and matching number of the divisor Cayley graph $G\ (Zn, D)$.

**Key words:** Cayley graph, Divisor Cayley graph, Edge cover, Edge covering number, Matching number.

**AMS Subject Classification (2010):** 68R10, 05C90, 05C70

## 1. Introduction

Nathanson [10] was the pioneer in introducing the concepts of number theory and thus paved way for the study of a new class of graphs called Arithmetic Graphs. The theory of groups provides an interesting and powerful abstract approach to the study of symmetries of various graphs and one such tool is the notion of Cayley graph $G(X, S)$, associated with a group $X$ and a symmetric subset $S$ of $X$, whose vertex set is $X$ and the edge set is the $\{(x, sx) \,/\, x \in X, s \in S\}$. Basically, the Cayley graphs are connected and vertex-transitive. If the group $(X, \cdot)$ is the additive group $(Z_n, \oplus)$ of integers $0,1,2, \ldots, n$ -1 modulo $n$, and the symmetric set $S$ is associated with some arithmetic function, then the Cayley graph may be treated as an arithmetic graph and such graphs are called arithmetic Cayley graphs. Madhavi [9] studied the arithmetic Cayley graphs associated with Euler-totient function, Quadratic residue function modulo a prime and the divisor function.

In a graph $G$, a vertex $v$ and an edge $e$ are said to cover each other if they are incident. An edge cover of a Graph $G$ is a set of edges covering all the vertices of $G$. The study of vertex domination, edge domination and other related concepts constitute the domination theory of graphs. It is said that a precise notion of a dominating set, that is present in the current literature is given by Berge [4], Ore [11]. Later Harary and Livington [8], Cockayne and Hedetniemi [6] and many others have contributed significantly to this theory.

In this paper, the author study the edge cover, edge covering number and matching number of the divisor Cayley graph $G(Z_n, D)$. We refer the reader for graph theoretic notions Bondy and

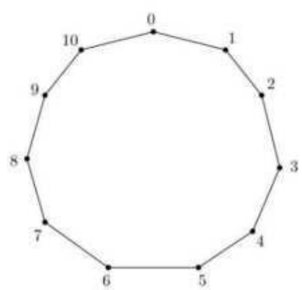
Murty [5] and Harary [7] and for number theoretic notions Apostol [2].

## 2. Divisor Cayley Graph

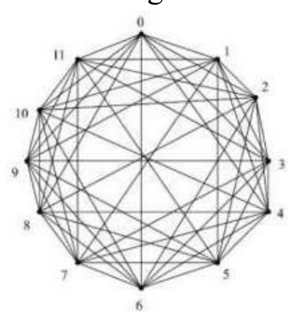
**Definition 2.1:** Let $n \geq 1$ be an integer and let S be the set of divisors of n. Then the set $S^* = \{s, n - s / s \in S\}$ is a symmetric subset of the group $(Z_n, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of $(Z_n, \oplus)$ associated with the above symmetric subset $S^*$ is called the **divisor of Cayley graph** and it is denoted by $G(Z_n, D)$. That is, the graph $G(Z_n, D)$ is the graph whose vertex set is $V = \{0, 1, 2, \ldots, n-1\}$ and the edge set E is the set of all ordered pairs of vertices x, y such that either $x - y \in S^*$, or , $y - x \in S^*$. Madhavi [6] introduced this graph and studied various properties of this graph. We list below some of them.

i. The graph $G(Z_n, D)$is $|S^*|$-regular, the number of edges in $G(Z_n, D)$is $\frac{n|S^*|}{2}$.
ii. The graph $G(Z_n, D)$ is Hamiltonian and hence connected.
iii. Degree of each vertex in $G(Z_n, D)$ is determined.
iv. The graph $G(Z_n, D)$ is not bipartite but Eulerian.
v. If $n$ is a prime, then the graph $G(Z_n, D)$ reduces to the outer Hamilton cycle.

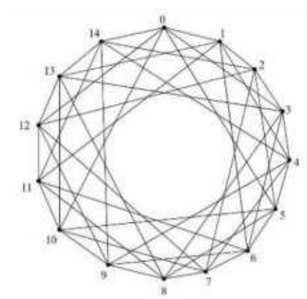
The graphs $G(Z_n, D)$ for $n =$ 11, 12 and 15 are given below.




$G(Z_{11}, D)$




$G(Z_{12}, D)$




$G(Z_{15}, D)$

## 3. Edge Cover of the Divisor Cayley Graph $G(Z_n, D)$

**Definition 3.1:** An **edge cover** of a graph $G$ is a set of edges covering all the vertices of $G$. A **minimum edge cover** is one with minimum cardinality. The number of edges in a minimum edge cover of $G$ is called the **edge covering number** of $G$ and it is denoted by $\beta'(G)$.

**Theorem 3.2**: If $n > 1$, the minimum edge covering of the divisor Cayley graph G is given by

(i) $\{(0,1), (2,3), \ldots, (n-4, n-3), (n-2, n-1)\}$, if n is even.
(ii) $\{(0,1), (2,3), \ldots, (n-3, n-2), (n-1, 0)\}$, if n is odd.

**<u>Proof :</u>**

(i) Suppose n is an even number. Consider the set of ordered pairs of vertices given by $F_1 = \{(0,1), (2,3), \ldots, (n-4, n-3), (n-2, n-1)\}$.

For each ordered pair $(2i, 2i+1)$, $0 \leq i \leq \frac{n-2}{2}$, $(2i+1) - 2i = 1 \in S^*$ so that $(2i, 2i+1)$ is an edge of $G(Z_n, D)$.

So $F_1$ is a set of edges in G. Further the edges in $F_1$ cover all the vertices of $G(Z_n, D)$. So $F_1$ forms an edge covering of $G(Z_n, D)$.

Furthermore, the end vertices of the edges in $F_1$ are distinct.

To show that $F_1$ is the minimum edge covering of $G(Z_n, D)$. Let us consider the edge set

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$F_1 - \{e_i\}$, where $e_i \in F_1$, then $e_i = (2i, 2i+1)$. Clearly, the vertices 2i, 2i+1 are not covered by the remaining edges of the edges set $F_1 - \{e_i\}$, so that $F_1 - \{e_i\}$ cannot form an edge covering of $G(Z_n, D)$.

Hence, $F_1$ is the minimum edge covering of $G(Z_n, D)$. Since n is even, the number of distinct pairs of distinct vertices of the form (2i, 2i+1), $0 \le i \le \frac{n-2}{2}$ so that the cardinality of the set $F_1$ is $\frac{n}{2}$.

(ii) Suppose n >1is an odd number. Consider the set ordered pairs of vertices given by
$F_2 = \{ (0,1), (2,3), \ldots, (n-3, n-2), (n-1, 0) \}$.

For each ordered pair (2i, 2i+1), $0 \le i \le \frac{n-1}{2}$, $(2i+1) - 2i = 1 \in S$, so that (2i, 2i+1) is an edge of $G(Z_n, D)$. So $F_2$ forms an edge covering of $G(Z_n, D)$.

To show that $F_2$ is the minimum edge covering of $G(Z_n, D)$, let us consider the edge set $F_2 - \{ e_i \}$. Then $e_i = (2i, 2i+1)$. Clearly, the vertices 2i, 2i+1 are not covered by the edge set $F_2 - \{ e_i \}$, so that $F_2$ is a minimum edge covering of $G(Z_n, D)$.

Since the n + 1 vertices 0, 1, 2, ……, n – 1 , 0 can be paired into $\frac{n+1}{2}$ distinct pairs of vertices ( 2i, 2i+1), $0 \le i \le \frac{n-1}{2}$, the cardinality of $F_2$ is $\frac{n+1}{2}$.

The following Corollary is immediate from the Theorem.

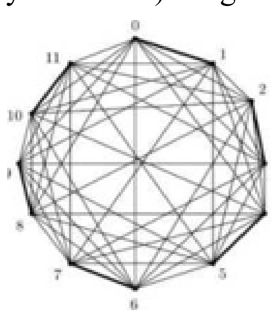
**Corollary : 3.3:**

If n > 1, the edge covering number of divisor Cayley graph $\beta^1(G(Z_n, D))$ is given by

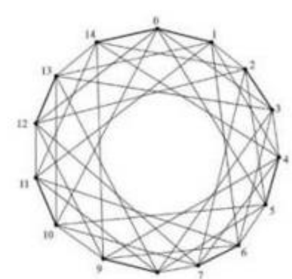
(i) $\frac{n}{2}$, if n is even.

(ii) $\frac{n+1}{2}$, if n is odd.

**Example 3.4:** The minimum edge covering sets of the graphs of $G(Z_{12}, D)$ and $G(Z_{15}, D)$ (denoted by bold lines) are given below.



Minimum edge covering of $G(Z_{12}, D)$



Minimum edge covering of $G(Z_{15}, D)$

For the graph $G(Z_{12}, D)$ , the minimum edge cover is given by {(0,1), (2,3), (4,5), (6,7), (8,9), (10,11)} and the edge covering number is $6 = \frac{12}{2}$

For the graph $G(Z_{15}, D)$,the minimum edge cover is given by {(0,1), (2,3), (4,5), (6,7), (8,9), (10,11), (12,13), (14,0)} and the edge covering number is $8 = \frac{15+1}{2}$.

## 4.The Edge Dominating set of the Divisor Cayley Graph $G(Z_n, D)$

**Definition: 4:**

A subset F of the edge set E in a graph G is an **edge dominating set if** each edge E not in F (that is in E – F) is adjacent to at least one edge in F.

The minimum cardinality among all edge dominating sets of G is called an **edge domination number** of G and is denoted by $\gamma^1(G)$.

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**Theorem : 4.1:**

The edge dominating set of the divisor Cayley graph G ($Z_n$, D), n > 2 is the set of edges.

(i) {(0,1), (2,3),….., ( n – 2, n – 1) } , if n is even.

(ii) {(1,2), (3, 4),….., ( n – 2, n – 1) } , if n is odd.

**Proof :**

(i) Let n be even. Consider the set of ordered pairs of vertices given by

$E_1$ = { (0,1), (2,3),….., ( n – 2 , n -1 ) } .

For each ordered pair (2i, 2i+1), $0 \leq i \leq \frac{n-2}{2}$, ( 2i+1) – 2i = 1 $\in S^*$, so that, (2i, 2i+1) is an edge of G ($Z_n$, D). So $E_1$ is a set of edges in G ($Z_n$, D). Clearly no two edges in $E_1$ are adjacent.

Let (r, s) $\in$ E – $E_1$. Then r ≥ 0 and s ≠ r + 1 Here two cases will arise, namely either r is even, or, r is odd.

**Case i :**

Suppose r is even. Then r = 2t, for some integer t ≥ 0 and (r, s) = (2t, s) Then the edge (2t, 2t+1) is in $E_1$ clearly it is adjacent with the edge (2t, s).

**Case ii:**

Suppose r is odd. Then r = 2t+1, for some integer t ≥ 0 and (r, s) = (2t +1, s). Then the edge (2t + 1, 2t) which is same as (2t, 2t+1) is in $E_1$ and this is adjacent with (2t+1, s). So, $E_1$ is an edge dominating set of G($Z_n$, D).

Let us now show that $E_1$ is the minimum edge dominating set of G($Z_n$, D). To see this, let us delete the edge (i, i+1) from $E_1$ and form the edge set $E_1^1 = E_1$ – { i, i+1 }. Now ( i, i+1) is not adjacent to any edge of the edge set $E_1^1$ , since any edge ( r, s) $\in E_1^1$ is such that r ≠ i , i + 1 and s ≠ i, i+1. So $E_1^1$ is not an edge dominating set of G($Z_n$, D). Hence, $E_1$ is the minimum edge dominating set of G($Z_n$, D).

(ii) Let n be odd. Consider the set of ordered pairs of vertices given by

$E_2$ = { (1,2), (3,4),….., ( n – 2 , n -1 ) } .

For each ordered pair ( 2i, 2i+1), $0 \leq i \leq \frac{n-1}{2}$, ( 2i+1) – 2i = 1 $\in S^*$ , so that ( 2i, 2i+1) is an edge of G($Z_n$, D). So $E_2$ is a set of edges in G($Z_n$, D).

Let (r, s) $\in$ E – $E_2$ . As in (i), we may assume r ≥ 1 and s ≠ r + 1 Here two cases will arise.

**Case i :**

Suppose r is odd. Then r = 2t+1, for some integer t > 0 and (r, s) = (2t +1, s). Then the edge (2t+1, 2t+2) is in $E_2$ and this is adjacent with the edge (2t+1, s).

**Case ii :**

Suppose r is even. Let r = 2t, for some integer t > 0. Then the edge (2t, s). Consider the edge (2t, 2t–1) which is same as (2t – 1, 2t) in $E_2$. This is adjacent with the edge (2t, s).

If r = 0, then the edge (0, s) is adjacent with (s – 1, s) in $E_2$, if s is even and it is adjacent with (s, s+1) in $E_2$, if s is odd.

Thus $E_2$ is an edge dominating set.

As in (i), we can see that $E_2$ is the minimum edge dominating set.

The following Corollary is immediate from the above Theorem.

**Corollary : 4.2:**

If n > 2, the edge domination number $\gamma^1$(G ($Z_n$, D)) is given by

i) $\frac{n}{2}$ , if n is even.

(ii) $\frac{n-1}{2}$, if n is odd.

**Proof :**

If n is even, the minimum edge dominating set of G ($Z_n$, D) is

$E_1$ = { (0,1), (2,3),….., ( n – 2 , n -1 ) }

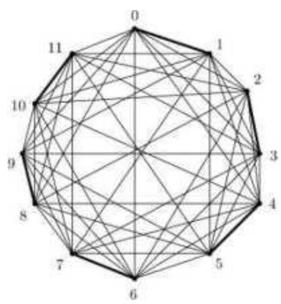
and this contains $\frac{n}{2}$ edges.

If n is odd, the minimum edge dominating set of G($Z_n$, D) is

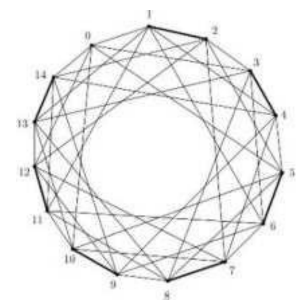
$E_2$ = { (1,2), (3,4),….., ( n – 2 , n -1 ) }

and this contains $\frac{n-1}{2}$ edges.

**Example 4.3:** The minimum edge dominating sets of the graphs of $G(Z_{12}, D)$ and $G(Z_{15}, D)$ (denoted by bold lines) are given below.




Minimum edge dominating set of $G(Z_{12}, D)$




Minimum edge dominating set of $G(Z_{15}, D)$

For the graph $G(Z_{12}, D)$, the minimum edge dominating set is given by $\{(0,1), (2,3), (4,5), (6,7), (8,9), (10,11)\}$ and the edge domination number is $6 = \frac{12}{2}$

For the graph $G(Z_{15}, D)$, the minimum edge dominating set is given by $\{(1,2), (3,4), (5,6), (7,8), (9,10), (11,12), (13,14)\}$ and the edge domination number is $7 = \frac{15-1}{2}$

## 5.The Matching Number of the Divisor Cayley Graph $G(Z_n, D)$

**Definition 5.1:** A **matching** $F$ of a graph $G$ is a subset of the edge set $E$ of $G$ such that no two edges of $F$ are adjacent.

**Definition 5.2:** A matching $F$ is called a **perfect matching** of $G$ if it covers all the vertices of the graph $G$

**Definition 5.3:** A maximal matching is a matching with maximum number of edges and the cardinality of a maximum matching is known as the **matching number**.

**Theorem 5.4:** The matching number of the divisor Cayley graph $G(Z_n, D)$, $n > 1$ is

(i) $\frac{n}{2}$, if $n$ is even and

(ii) $\frac{n-1}{2}$, if $n$ is odd.

Proof:

(i) Let n be even. Consider the set of ordered pairs of vertices given by

$F_1$ = { (0,1), (2,3),….., ( n – 2 , n –1 ) }.

For each ordered pair ( 2i, 2i+1), $0 \le i \le \frac{n-2}{2}$, ( 2i+1) – 2i = 1 $\in S^*$ , so that ( 2i, 2i+1) is an edge of G($Z_n$, D). So $F_1$ is a set of edges in G($Z_n$, D).

It is easy to see that no two edges in $F_1$ are adjacent. So, $F_1$ is the matching of the graph G($Z_n$, D).

Let us now show that $F_1$ is the maximal matching of $G(Z_n, D)$. To see this let us consider the edge set $\{ (0,1), (2,3), \ldots, (n-2, n-1) \} \cup \{ r, s\}$, where $r \geq 0$ and $s \neq r + 1$.

Suppose r is even. Then $r = 2t$, for some integer $t \geq 0$ and $(r, s) = (2t, s)$. Then the edge $(2t, 2t+1)$ is in $F_1$ and is adjacent with $(2t, s)$.

Suppose r is odd, then $r = 2t+1$, for some integer $t \geq 0$ and $(r, s) = (2t+1, s)$. Then the edge $(2t+1, 2t)$, which is same as $(2t, 2t+1)$, is in $F_1$ and is adjacent with $(2t+1, s)$.

So, the edge set $\{ (0,1), (2,3), \ldots, (n-2, n-1)\} \cup \{r, s\}$ where $r \geq 0$ and $s \neq r+1$ is not matching of the graph $G(Z_n, D)$. Hence, the edge set $\{ (0,1), (2,3), \ldots, (n-2, n-1)\}$ is a maximal matching of the graph $G(Z_n, D)$.

In this case the cardinality of a maximal matching of the graph $G(Z_n, D)$ is $\frac{n}{2}$ .

(ii) Let n be odd. Consider the set of ordered pairs of vertices given by

$$F_2 = \{ (0,1), (2,3), \ldots, ( n-3 , n-2 ) \}.$$

For each ordered pair $(2i, 2i+1)$, $0 \leq i \leq \frac{n-3}{2}$, $(2i+1) - 2i = 1 \in S^*$, so that $(2i, 2i+1)$ is an edge of $G(Z_n, D)$. So $F_2$ is a set of edges in $G(Z_n, D)$.

By inspection, it is easy to see that no two edges in $F_2$ are adjacent. So, $F_2$ is the matching of the graph $G(Z_n, D)$.

To show that $F_2$ is a maximal matching of the graph $G(Z_n, D)$, let us consider the edge set $\{ (0,1), (2,3), \ldots, (n-3, n-2) \} \cup \{ r, s\}$ where $r \geq 0$ and $s \neq r + 1$.

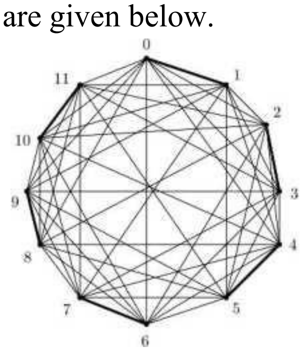
Suppose r is even. Then $r = 2t$, for some integer $t \geq 0$ and $(r, s) = (2t, s)$. Then the edge $(2t, 2t+1)$ is in $F_2$ and is adjacent with $(2t, s)$.

Suppose r is odd, then $r = 2t+1$, for some integer $t \geq 0$ and $(r, s) = (2t+1, s)$. Then the edge $(2t+1, 2t)$, which is same as $(2t, 2t+1)$ is in $F_2$ and is adjacent with $(2t+1, s)$.

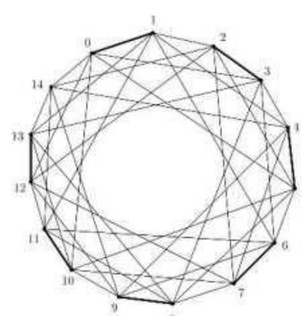
If $r = n-1$, then the edge $(n-1, s)$ is adjacent with $(s-1, s)$ in $F_2$, if s is even and it is adjacent with $(s, s+1)$ in $F_2$, if s is odd. So, the edge set $\{ (0,1), (2,3), \ldots, (n-3, n-2 )\} \cup \{r, s\}$, where $r \geq 0$ and $s \neq r+1$ is not matching of the graph $G(Z_n, D)$. Hence, the edge set $\{(0, 1), (2,3), \ldots\ldots\ldots, (n-3, n-2 )\}$ is a maximal matching of the graph $G(Z_n ,D)$.

**The cardinality of a maximal matching of the graph $G (Z_{n, D})$ is $\frac{n}{2}$**

**Example 5.5:** The maximal matching sets of the graphs of $G(Z_{12}, D)$ and $G(Z_{15}, D)$ (denoted by bold lines) are given below.



Maximal matching set of $G(Z_{12}, D)$



Maximal matching set of $G(Z_{15}, D)$

For the graph $G(Z_{12}, D)$, the maximal matching set is given by $\{(0,1), (2,3), (4,5), (6,7), (8,9), (10,11)\}$ and the matching number is $6 = \frac{12}{2}$

For the graph $G(Z_{15}, D)$, the maximal matching set is given by $\{(0,1), (2,3), (4,5), (6,7),$

$(8,9), (10,11), (12,13)\}$ and the matching number is $7 = \frac{15-1}{2}$

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